

Exam 2, Module 7, Code 201600270
Discrete Structures & Efficient Algorithms
Friday April 7, 2017, 13:45 - 16:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4) per topic (L&M,ALG,DM). Also if you cannot solve a part of an question you may use that result in subsequent parts of the question.

This exam consists of three parts, with the following (estimated) times:

Languages & Machines (L&M)	1h	(30 points)
Algebra (ALG)	1h 40 min	(50 points)
Discrete Mathematics (DM)	20 min	(10 points)

Total of 30+50+10=90 points. Including 10 bonus points that makes 100 points. Your exam grade is the total number of points divided by 10.

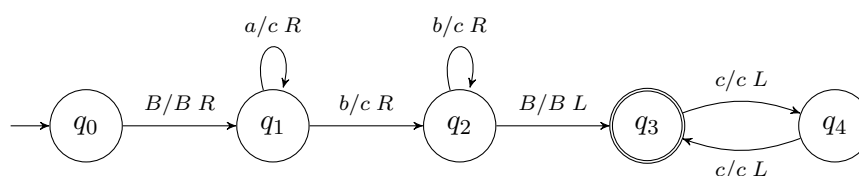
Please use a new sheet of paper for each part (L&M/ALG/DW)!

Languages & Machines

1. (a) (6 points) Transform the following contextfree grammar G step by step to an equivalent grammar G' in Chomsky Normal Form.

$$G = \begin{cases} S \rightarrow aA \\ A \rightarrow \lambda \mid B \mid aA \\ B \rightarrow c \mid Bc \end{cases}$$

- (b) (6 points) Let $w = aacc$. Apply the CYK-algorithm (after Cocke-Younger-Kasami) to decide whether $w \in \mathcal{L}(G')$. Provide a derivation tree for w as well.
2. (6 points) Consider the contextfree language $L = \{a^{2i} b^i c \mid i \geq 0\}$. Give a *deterministic* PDA (stack automaton) for L . Provide a *short* explanation.
3. (6 points) Which language is *decided* by the following Turing Machine? (only q_3 is accepting)? Explain your answer *shortly*.



4. (6 points, every wrong answer costs 2 points) Indicate for each of the following statements if they are TRUE or FALSE. (No explanation required).
- (a) Every contextfree grammar (CFG) has a Turing Machine (TM) accepting the same language.
- (b) Every contextfree grammar (CFG) has an equivalent extended PDA with two states.

- (c) The class of contextfree languages is closed under complement.
 - (d) The class of contextfree languages is closed under union.
 - (e) To every PDA there exists a equivalent deterministic PDA.
 - (f) To every TM there exists an equivalent deterministic TM.
 - (g) The language of (encoded) terminating Turing Machines is not recursief, but it is recursive enumerable.
 - (h) Given a grammar G in Chomsky Normal Formal Form and a word w , one can decide in polynomial time whether $w \in \mathcal{L}(G)$.
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Algebra

Do not spend too much time on the individual parts. If you cannot solve a part, you may still use the result of that part for the subsequent part(s).

5. Let G be the set of matrices given by:

$$G = \left\{ \begin{bmatrix} \alpha & \beta \\ 2\beta & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{Z}_3 \quad \alpha^2 + \beta^2 \neq 0 \right\}.$$

On G we consider the operation matrix multiplication.

- (a) Show that G with matrix multiplication forms a group.
- (b) Let $\mathbb{F} = \mathbb{Z}_3[x] / \langle x^2 + 1 \rangle$. Show that $\phi : G \rightarrow \mathbb{F} \setminus \{0\}$ defined by

$$\phi \left(\begin{bmatrix} \alpha & \beta \\ 2\beta & \alpha \end{bmatrix} \right) = \alpha + \beta x + \langle x^2 + 1 \rangle$$

is a group isomorphism from G to the multiplicative group of the field \mathbb{F} .

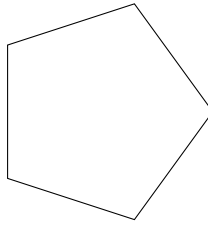
6. Given the permutations:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}$$

Write α, β and $\beta\alpha$ as:

- (a) Product of disjoint cycles.
- (b) Product of 2-cycles.
- (c) Determine the order of α .

7. Use Burnside's theorem to determine the number of different ways in which the edges of a regular pentagon (see figure), made of copper wire, can be colored using two colors.



8. (a) Let $a(x) = x^2 + a_1x + a_0 \in \mathbb{Z}_2[x]$. Determine all values of $a_0, a_1 \in \mathbb{Z}_2$ for which $a(x)$ is irreducible.

Let $p(x) = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x]$.

- (b) Prove that $p(x)$ is irreducible.

Let $\mathbb{F} = \mathbb{Z}_2[x]/\langle p(x) \rangle$.

- (c) Is \mathbb{F} a field?

- (d) What is the number of elements of \mathbb{F} ?

Points: **Ex 5**, a: 6, b: 6, **Ex 6**: a: 5, b: 5, c: 4, **Ex 7**: 10, **Ex 8**: a: 3, b: 4, c: 3, d: 4.